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Matrix Method for Eigenstructure Assignment: The Multi-Input Case with Application

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The eigenvalue and eigenstructure assignment procedure has found application in a wide variety of control problems. In this paper a method for assigning eigenstructure to a linear time invariant multi-input system is proposed. The algorithm determines a matrix that has eigenvalues and eigenvectors at the desired locations. It is obtained from the knowledge of the open-loop system and the desired eigenstructure. Solution of the matrix equation, involving unknown controller gains, open-loop system matrices, and desired eigenvalues and eigenvectors, results in the state feedback controller. The proposed algorithm requires the closed-loop eigenvalues to be different from those of the open-loop case. This apparent constraint can easily be overcome by a negligible shift in the values. Application of the procedure is illustrated through the offset control of a satellite supported, from an orbiting platform, by a flexible tether.

Nomenclature

A, B, F	= $A \in R^{n\times n}$, $B \in R^{n\times m}$, $F \in R^{m\times n}$, respectively
A_c	= $A_c \in R^{n\times n}$
B_1	= first mode for longitudinal tether deflection
C_1	= first mode for lateral tether deflection
d_{py}	= offset of tether attachment point (along Y_p axis) from platform center of mass, c.m.
d_{pz}	= offset of tether attachment point (along Z_p axis) from platform center of mass, c.m.
GM	= gravitational constant
I	= identity matrix
I_{xx_p}	= moment of inertia of platform about X_p axis
I_{yy_p}	= moment of inertia of platform about Y_p axis
I_{zz_p}	= moment of inertia of platform about Z_p axis
L	= length of tether
M_x	= moment applied at center of mass of platform to control α_p
m_p	= mass of platform
m_r	= mass of reel located at tether attachment point
m_s	= mass of subsatellite
m_0	= $m_r + \rho_t L + m_s$
O	= null matrix
R_c	= orbital radius
v_i	= ith eigenvector of $A_c \in C^n$
x, u	= state vector $\in R^n$ and control input vector $\in R^m$, respectively
α_p	= platform pitch angle
α_t	= tether pitch angle
θ	= true anomaly
ρ_t	= mass per unit length of tether
λ_i	= ith eigenvalue of $A_c \in C$

Dots denote differentiation with respect to time.

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Introduction

ONE of the fundamental concepts of control theory is that of feedback. The main objectives of feedback in practice (reported in most classical textbooks on control theory) are to i) ensure and improve stability characteristics of the system, ii) alter the system transient response, iii) reduce sensitivity of the system to modeling errors, and iv) improve the system's capability to reject disturbances and attenuate noise.

In the context of the "modern" control theory, a number of methodologies have evolved, to address these issues, in the last 30 years, such as a) eigenvalue and eigenstructure (eigenvalue/eigenvector) assignment, b) linear quadratic regulator (LQR) and linear quadratic Gaussian (LQG) type of optimal control, c) robust control procedures accounting for the parameter and model uncertainties, d) adaptive control, etc.

Among these, the eigenvalue assignment procedure is, of course, the simplest. Hence the procedure has become central to a large number of control system problems. Its importance has led to numerous studies resulting in a vast body of literature, some of which has become classical textbook material.[1,2] Rosenbrock,[3] Kalman,[4] and Wonham[5] rank among the pioneering researchers in the area of eigenvalue assignment. Rosenbrock addressed the issue for the single-input case, whereas Wonham extended the result from single-input to multi-input systems. Kalman was the first to develop a methodology for construction of irreducible realization of impulse response function of arbitrary dimensional systems.

In the problems of pole placement using full state feedback, the primary concern of the investigators during the 1960s and early 1970s was the "stabilization" of the system as opposed to shaping of the transient response. Also in the multi-input case, the feedback gain matrix for a given set of eigenvalues is not unique. Freedom offered by the state feedback beyond specification of the closed-loop eigenvalues was first identified by Moore[6] in 1976. In Moore's work the necessary and sufficient conditions for the existence of a state feedback controller, which yields prescribed eigenvalues and eigenvectors, were derived for the case where the desired closed-loop eigenvalues are distinct. Since 1967, when Wonham's paper[5] appeared, there have been literally hundreds of papers written concerning eigenstructure assignment and its application.[7–18] References 7 and 8 deal with the eigenstructure assignment issue in conjuction with the state feedback whereas Andri et al.[9] addressed the issue using the state, output, and constrained output feedback.

From a numerical standpoint most of the algorithms developed so far can be categorized as[16] i) the implicit QR method,[10,11] ii) pole assignment via Sylvester's equation,[12] iii) solution using the real Shur form,[13] iv) a singular-value decomposition (SVD) procedure,[14] and v) the approach involving orthogonal reduction to a Block-Hessenberg form with simple linear recursion.[15,16] Note that SVD-based method is an iterative eigenstructure method, whereas others are direct eigenvalue assignment procedures. Varga[13] uses the real Schur form, which is relatively expensive to compute and, in an ill-conditioned eigenproblem, may introduce errors that are avoided by employing the staircase-type reduction. The methods listed above (except v) use some form of staircase reduction. Method v is applicable to well-conditioned problems only. Nichols[17] has described a measure of robustness for the closed-loop system and indicated techniques for its check and improvement; Datta and Datta[18] have discussed application of a parallel algorithm.

This paper proposes an algorithm for the eigenstructure assignment of a multi-input linear time-invariant (LTI) system. This is an extension of the algorithm, proposed in Ref. 19, from the single-input to multi-input systems. The algorithm consists of characterizing the space in which the achievable eigenvectors lie. If the specified closed-loop eigenvectors do not belong to the achievable eigenspace, then a set of assignable eigenvectors can be computed. This is followed by a similarity transformation of the closed-loop system states. Finally, some algebraic manipulation results in a closed-form expression for the state feedback controller. The algorithm requires the desired closed-loop eigenvalues to be different from those of the open-loop system to avoid singularity leading to a numerical problem. Such possibility can be avoided by placing the closed-loop eigenvalues near (but not exactly at) the open-loop poles whenever required. The proposed direct procedure for determination of the feedback gain, in contrast to iterative approaches reported in the literature, is indeed attractive. It does not involve determination of eigenvalues or singular values, and hence the associated computational problem is completely eliminated. It is also important to emphasize that all possible types and combinations of eigenvalues (real distinct, complex, and repeated) are accounted for in the investigation.

Preliminaries and Problem Statement

Consider a controllable LTI system given by

$$\dot{x} = Ax + Bu \tag{1}$$

where $A \in R^{n\times n}$ is the open-loop system matrix, $B \in R^{n\times m}$ the control influence matrix, $x \in R^n$ the state vector, and $u \in R^m$ the control input vector. With the state feedback, $u = Fx$, the closed-loop system can be expressed as

$$\dot{x} = A_c x \tag{2}$$

where

$$A_c = A + BF \tag{3}$$

is a closed-loop system matrix and F is a matrix of controller gains $\in R^{m\times n}$. A solution representing the free response of Eq. (2) depends on three quantities[9]: i) eigenvalues, which determine the decay/growth rate of the response; ii) eigenvectors, which determine the shape of the response; and iii) initial conditions, which determine the degree to which each mode will participate in the free response. Thus it is quite apparent that if feedback is to be used to alter the system transient response, eigenvector selection must be considered as well as eigenvalue placement. Though the initial conditions affect the system's transient response, they are not important for linear systems.

The structure of the eigenvalue problem for the closed-loop system given in Eq. (2) depends on the types of eigenvalues. For distinct roots of A_c, the eigenvalue problem is

$$\lambda_i v_i = A_c v_i \tag{4}$$

where the λ_i are closed-loop eigenvalues and the v_i the corresponding eigenvectors. Substituting Eq. (3) into Eq. (4) and rearranging the terms, v_i can be expressed as

$$v_i = (\lambda_i I - A)^{-1} B m_i \tag{5}$$

where

$$m_i = F v_i \tag{6}$$

For repeated roots, the generalized eigenvalue problem for the above-mentioned closed-loop system can be expressed as

$$\lambda_j v_j = A_c v_j \tag{7a}$$

$$\lambda_j v_{i+1} + v_i = A_c v_{i+1}, \qquad j \le i < k \tag{7b}$$

where k is the multiplicity of the eigenvalue λ_j. As in the case of the distinct eigenvalues, Eq. (7) can be simplified to

$$v_j = (\lambda_j I - A)^{-1} B m_j \tag{8a}$$

$$v_{i+1} = (\lambda_j I - A)^{-1} B m_{i+1} - (\lambda_j I - A)^{-1} v_i, \qquad j \le i < k \tag{8b}$$

where the m_i are defined as in Eq. (6). In Eqs. (5), (8a), and (8b), it is assumed that the desired poles of the system do not coincide with the eigenvalues of A so that the inverse $(\lambda_i I - A)^{-1}$ exists. If this is not the case, the desired eigenvalues may be changed by an infinitesimal amount to avoid the singularity problem without affecting the system dynamics.

With these basic concepts concerning the eigenstructure of the system, the problem under consideration can be stated as follows: Given a controllable LTI system (A, B), a self-conjugate set of scalars $\{\lambda_i\}$, $i = 1, 2, \ldots, n$, and a corresponding self-conjugate set of n-vectors $\{v_i^d\}$, $i = 1, 2, \ldots, n$, find a real $m \times n$ matrix K such that the eigenvalues of $A + BK$ are precisely those of the set of scalars $\{\lambda_i\}$ with the corresponding eigenvector set $\{v_i^d\}$. (A set is said to be self-conjugate if for a quantity that is a member of the set its complex conjugate is also a member.)

In general, for an arbitrary linear system, it is not always possible to assign the set $\{v_i^d\}$ exactly. Under such situations, another set of vectors $\{v_i^a\}$ can be assigned as the eigenvectors such that it is the projection of the set $\{v_i^d\}$ on the achievable eigenspace. This issue is addressed in the next section.

Eigenstructure Assignment

This section presents an algorithm for eigenstructure assignment by state feedback for systems described by Eq. (1). The algorithm is presented in three steps. The total specification of v_i^d is considered first followed by the characterization of the best possible achievable eigenvectors. Finally, computation of the state feedback gain is presented.

Total Specification of v_i^d

Consider Eq. (5), which is valid for the distinct eigenvalues of the closed-loop system matrix A_c: $v_i = (\lambda_i I - A)^{-1} B m_i$. It implies that the eigenvector v_i must be in the subspace spanned by the columns of $(\lambda_i I - A)^{-1} B$. This subspace has dimension m, which is equal to the rank of B, i.e., the number of independent control variables. Therefore, the number of independent control variables available determines the dimension of the subspace in which the achievable eigenvectors must reside. The orientation of this subspace is determined by the open-loop parameters described by A, B, and the desired closed-loop eigenvalues λ_i. So it can be concluded that if the desired eigenvector v_i^d lies precisely in the subspace spanned by the columns of $(\lambda_i I - A)^{-1} B$, then it will be achieved exactly.[9]

Now consider the eigenvectors corresponding to the repeated eigenvalue λ_j as in Eqs. (8a) and (8b):

$$v_j = (\lambda_j I - A)^{-1} B m_j$$

$$v_{i+1} = (\lambda_j I - A)^{-1} B m_{i+1} - (\lambda_j I - A)^{-1} v_i, \qquad j \le i < k$$

where k is the multiplicity of the eigenvalue λ_j. The condition of assignability of v_j is the same as that of the distinct eigenvalues. For deriving the assignability condition for v_{i+1}, $j \le i < k$, the expression for v_{i+1} can be rewritten as

$$v_{i+1} + (\lambda_j I - A)^{-1} v_i = (\lambda_j I - A)^{-1} B m_{i+1}, \qquad j \le i < k$$

From this equation it can be concluded that if the eigenvector v_{i+1}, $j \le i < k$, is chosen in such a way that the vector $\tilde{v}_{i+1} \triangleq v_{i+1} + (\lambda_j I - A)^{-1} v_i$ lies precisely in the subspace spanned by the columns of $(\lambda_j I - A)^{-1} B$, then it will be achieved exactly.

Characterization of Achievable Eigenvectors

In general, however, a desired eigenvector v_i^d may not reside in the prescribed subspace and hence cannot be achieved. Instead a "best possible" choice of an achievable eigenvector can be made. This achievable eigenvector v_i^a is the projection of v_i^d onto the subspace spanned by the columns of $(\lambda_i I - A)^{-1} B$. The eigenvector v_i^a can be obtained in such a way that the norm of the vector difference between the desired and best possible achievable eigenvector,[9] i.e.,

$$J = \|v_i^d - v_i^a\|^2 \tag{9}$$

is minimized. To this end, the achievable eigenvector can be expressed as

$$v_i^a = (\lambda_i I - A)^{-1} B z_i$$

where z_i is so selected as to minimize J. Minimization of J results in

$$v_i^a = P_i (P_i^T P_i)^{-1} P_i^T v_i^d \tag{10}$$

where $P_i \triangleq (\lambda_i I - A)^{-1} B$. It should be pointed out that the matrix product $P_i^T P_i$ can be ill conditioned, and hence care should be taken while inverting it. Sometimes it may be possible to overcome this difficulty by defining new control inputs, thereby scaling the original control inputs and hence scaling the matrix B. In that case, the scaling matrix has to be nonsingular; i.e., it preserves the rank of B.

Based on the previous analysis, the following comments can be made:

i) If a desired eigenvector v_i^d is nearly orthogonal to the subspace spanned by the columns of $(\lambda_i I - A)^{-1} B$, there is a little chance of affecting the system response by v_i^a as computed above. The reverse is true when v_i^d lies in the subspace.

ii) For a single-input system B is simply a column vector; hence there is no possibility of affecting the transient response by altering the eigenvectors. It should be pointed out that only one element of each eigenvector can be specified, which will not alter the transient respone.

iii) If a larger number of entries in the eigenvectors are to be assigned, more independent control variables are necessary. This follows from the fact that the dimension of the subspace generated by $(\lambda_i I - A)^{-1} B$ is m. Thus, to increase m, the dimension and rank of B must be increased. When the rank of B is n, the entire eigenvector can be completely specified.

iv) In case of ill-conditioned B, the achievable eigenspace corresponding to the scaled inputs is the same as the original achievable eigenspace. So the scaling of the control inputs does not affect the achievable eigenspace.

Computation of State Feedback Controller Gain

The purpose of the controller design algorithm is to compute the state feedback gain vector F so that the closed-loop system described by

$$\dot{x} = (A + BF)x = A_c x$$

will have eigenvalues and eigenvectors at the desired locations.

Now consider the closed-loop system modal matrix T, whose columns are the assignable eigenvectors, i.e.,

$$T = [v_1^a \quad v_2^a \quad \cdots \quad v_n^a]$$

Transformation of the closed-loop system by the similarity relation $z = Tx$ results in

$$\dot{z} = A_d z \tag{11a}$$

with

$$A_d = T A_c T^{-1} \tag{11b}$$

where the transformed system matrix A_d is the Jordan canonical form of A_c. In the above equation, T is a known matrix because all the assignable eigenvectors are known. The matrix A_d can be obtained from the knowledge of the desired closed-loop eigenvalues. Simplification of Eq. (11b) results in

$$A_c = A + BF = T^{-1} A_d T \tag{12}$$

or

$$BF = T^{-1} A_d T - A \tag{13}$$

which can be solved for F (the state feedback controller):

$$F = [B^T B]^{-1} B^T [T^{-1} A_d T - A] \tag{14}$$

It might be appropriate to comment on two aspects of importance associated with development of the algorithm. The first issue concerns the real or complex character of the matrices involved. In general, these matrices (particularly T and A_d) can have complex entries. Since dealing with complex matrices demands time and effort, it is convenient to convert the complex Eq. (12) into a real matrix equation. This can be achieved by multiplying both sides of Eq. (12) by a nonsingular matrix[6]

$$L = \begin{bmatrix} L_1 & O \\ O & L_2 \end{bmatrix} \tag{15}$$

where

$$L_1 = I_p$$

$$L_2 = \begin{bmatrix} 0.5 & -0.5i & 0 & 0 & \cdots \\ 0.5 & 0.5i & 0 & 0 & \cdots \\ 0 & 0 & 0.5 & -0.5i & \cdots \\ 0 & 0 & 0.5 & 0.5i & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \in C^{(n-p)\times(n-p)}$$

In the above equation, without loss of generality, it is assumed that in the set of desired eigenvalues, the first p elements are real and the rest are complex.

The second aspect to be addressed is the possibility of an ill-conditioned matrix product $[B^T B]$. This situation arises when the condition number of B is very high. In such situations care will have to be exercised while inverting the matrix product $[B^T B]$. The difficulty involved in the matrix inversion can be overcome by scaling the control input as explained below.

Consider the open-loop system

$$\dot{x} = Ax + Bu,$$

which can be redefined as

$$\begin{aligned} \dot{x} &= Ax + B_m M u \\ &= Ax + B_m u_m \end{aligned} \tag{16}$$

where

B_m = modified control influence matrix having smaller condition number, BM^{-1}

M = nonsingular scaling matrix $\in R^{m\times m}$

u_m = modified control input, Mu

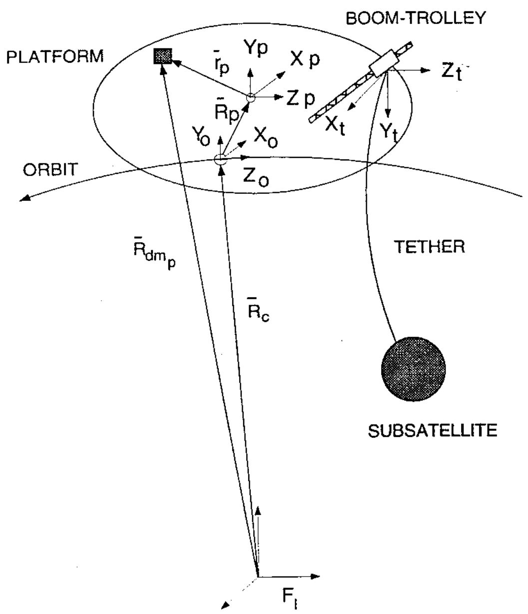

Fig. 1 Schematic diagram of TSS showing coordinate systems.

The scaling matrix M is so selected as to make B_m well conditioned; i.e., $B_m^T B_m$ is easily invertible. Since the matrix B_m is well conditioned, the state feedback controller $u_m = F_m x$ can be designed for the system in Eq. (16), which can be translated back to the original control input

$$u = M^{-1} u_m = M^{-1} F_m x$$
$$= Fx$$

where $F = M^{-1} F_m$ is the controller gain matrix for the original system.

Application to Tethered Satellite Systems

In the early stages of space exploration, satellites tended to be relatively small, simple in design, and essentially rigid. However, the passage of time clearly suggests a trend toward larger and necessarily flexible spacecraft. Communication satellites with large solar panels (Olympus, 25 m); the Space Shuttle based Solar Array Flight Experiment (SAFE, 1984, 31 m); NASA's proposed experiment as part of the Control Structure Interaction (CSI) study involving 40 m or longer flexible mast; the Space Station Freedom, extending to 115 m; and the Space Shuttle supported Tethered Satellite System (TSS) deployed to 20 km illustrate the point. This class of systems is governed by lengthy, highly nonlinear, nonautonomous and coupled equations of motion. One of the challanges of the 1990s is the resolution of formidable control/structure interaction problems.

As mentioned before, a variety of control procedures have evolved to address this issue. Among these the eigenstructure assignment, which is the subject of the present work, is one of the earliest control algorithms used to shape the transient behavior of the system.

In this paper, control of a space platform based TSS (Fig. 1) is considered to assess the validity of the proposed algorithm. The system consists of an orbiting rigid space platform to which a rigid point mass subsatellite is attached by an elastic tether. The tether attachment point at the platform end can be moved to control the tether swing motion (referred to as the offset control strategy). Inplane dynamics of the TSS with both longitudinal and inplane transverse flexibility of the tether are considered in the present study.

This section consists of three major steps: development of the mathematical model representing a general TSS undergoing planar motion, derivation of a simpler linear model for controller design purpose, and design of the controller and its implementation on the general model.

Development of Mathematical Model

The starting point for the development of a mathematical model is the selection of reference frames that can be used to define the motion of different elements of the system. The TSS under consideration (Fig. 1) consists of four different subsystems (also called domains): platform, boom-trolley, tether, and subsatellite. As shown in Fig. 1, the frame $F_p(x_p, y_p, z_p)$ is attached to the center of mass of the rigid platform and defines the libration of the platform. The boom-trolley arrangement is used to move the tether attachment point on the platform and is considered as a point mass. The frame $F_t(x_t, y_t, z_t)$ is attached to the tether at the platform end and is used to define the motion (both rigid and flexible) of any elemental mass of the tether. The fourth domain, the subsatellite, is considered as a point mass. The frame F_p is so oriented that its axes coincide with the principal axes of the platform. The frame attached to the tether, i.e., F_t, is oriented as shown in Fig. 1.

Two additional reference frames are required to completely define the kinematics of any mass element in the system: inertial frame F_I located at the center of the earth and the orbital frame $F_o(x_o, y_o, z_o)$. The origin of the orbital frame is located at the instantaneous center of mass of the system and follows the Keplerian orbit. The orbital frame is so oriented that the y_o axis is along the local vertical and points away from the earth, the z_o axis is along the local horizontal and points toward the direction of motion of the system, and the x_o axis is along the orbit normal and completes the right-handed triad.

The position vector $\bar{R}_{dm_i}$ of any elemental mass in the ith domain, with respect to the inertial frame, can be expressed as

$$\bar{R}_{dm_i} = \bar{R}_c + \bar{R}_i + \bar{r}_i, \qquad i = p, t$$

where

$\bar{R}_c$ = orbital radius
$\bar{R}_p$ = position vector of origin of F_p with respect to F_o
$\bar{R}_t$ = position vector of origin of F_t with respect to F_o
$\bar{r}_i$ = position vector of elemental mass in domain i with respect to F_i

The subscripts p and t refer to platform and tether, respectively.

The position vector as expressed above can be completely defined by discretizing the flexible tether using the "assumed mode" method. Once the position vector is defined, the inertial velocity can be obtained, and hence the kinetic, potential, and strain energies can be expressed in terms of the system variables. Application of the Lagrange procedure results in the equation of motion.

Derivation of Linear Model for Controller Design

Enormous amount of effort, without any significant gain, will be required if the complete model is used for the controller design. So a simplified model is obtained based on the dynamical simulation of the complete nonlinear flexible system. Simulation is carried out for a tether length of 500 m with the following mass and inertia parameters

$$m_p = 100{,}000 \text{ kg}, \qquad I_{xx_p} = 20.33 \times 10^7 \text{ kg-m}^2$$
$$m_r = 10 \text{ kg}, \qquad I_{yy_p} = 8.33 \times 10^7 \text{ kg-m}^2$$
$$m_s = 100 \text{ kg}, \qquad I_{zz_p} = 12.00 \times 10^7 \text{ kg-m}^2$$

The axial stiffness of the tether (EA) is taken to be 10^4 N, the diameter of the tether is 1.4×10^{-4} m, and the mass density of the tether material is 0.29×10^{-4} kg/m[20].

The initial conditions are $\alpha_p(0) = 5$ deg, $\alpha_t(0) = 5$ deg, $B_1(0) = 0.0084$ m, $C_1(0) = 0.2$ m, and $\dot{\alpha}_p(0) = \dot{\alpha}_t(0) = \dot{B}_1(0) = \dot{C}_1(0) = 0$.

From the simulation result (Fig. 2), it is clear that the rigid-body rotation of the tether is essentially unaffected by the tether flexibility. The same is true for the platform pitch response. This can be expected because for $d_{py} = 20$ m and $d_{pz} = 0$, which is considered in the present simulation, the coupling between the platform and the tether motion is very small. As seen from the figure, the longitudinal flexible response consists of three frequencies. The lowest frequency is due to the coupling between the rigid-body dynamics and the flexible motion, whereas the highest frequency corresponds to the flexibility itself. The third frequency is the result of the coupling between the longitudinal and transverse oscillations of the tether.

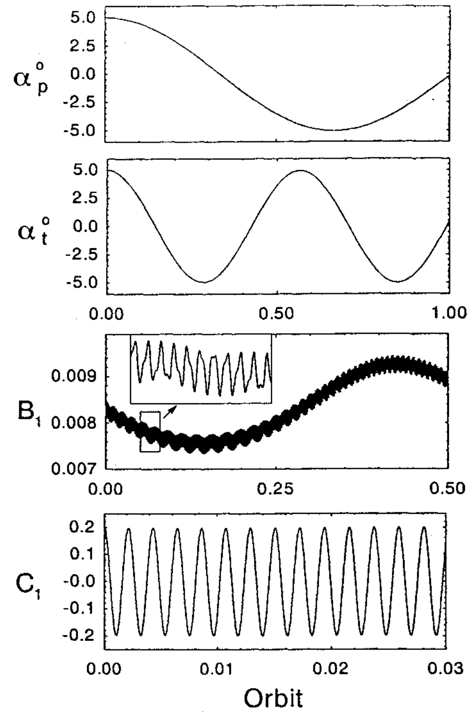

Fig. 2 Numerical parametric response study using exact nonlinear, nonautonomous and coupled equations of motion: $L = 500$ m, $d_{py} = 20$ m, and $d_{pz} = 0$.

With these observations of the system dynamics the model for the controller design can be obtained as follows.

Since the purpose of the controller is to regulate the rigid-body rotations, based on the simulation results, it can be concluded that the rigid-body model is sufficient for the controller design. On eliminating the flexible terms from the general model, the governing equations of motion of the rigid-body dynamics can be expressed as follows:

For the α_p equation

$$\begin{aligned}
&[I_{xx_p} + m_0(d_{py}^2 + d_{pz}^2)]\ddot{\alpha}_p \\
&+ m_s L[d_{pz}\sin(\alpha_t - \alpha_p) + d_{py}\cos(\alpha_t - \alpha_p)]\ddot{\alpha}_t \\
&+ m_0 d_{py}\ddot{d}_{pz} + [I_{xx_p} + m_0(d_{py}^2 + d_{pz}^2)]\ddot{\theta} \\
&+ m_s L[d_{pz}\sin(\alpha_t - \alpha_p) + d_{py}\cos(\alpha_t - \alpha_p)]\ddot{\theta} \\
&+ 2m_0 d_{pz}\dot{d}_{pz}(\dot{\alpha}_p + \dot{\theta}) \\
&- m_s L(\dot{\alpha}_t + \dot{\theta})^2[d_{py}\sin(\alpha_t - \alpha_p) - d_{pz}\cos(\alpha_t - \alpha_p)] \\
&+ \frac{\mathrm{GM}}{2R_c^3}\{3m_0[(d_{py}^2 - d_{pz}^2)\sin(2\alpha_p) + 2d_{py}d_{pz}\cos(2\alpha_p)] \\
&+ 2m_s L[d_{py}\sin(\alpha_t - \alpha_p) - d_{pz}\cos(\alpha_t - \alpha_p)] \\
&+ 6m_s L\cos(\alpha_t)[d_{py}\sin(\alpha_p) + d_{pz}\cos(\alpha_p)] \\
&+ 3(I_{zz_p} - I_{yy_p})\sin(2\alpha_p)\} = M_x
\end{aligned}$$

For the α_t equation

$$\begin{aligned}
&m_s L^2\ddot{\alpha}_t + m_s L[d_{pz}\sin(\alpha_t - \alpha_p) + d_{py}\cos(\alpha_t - \alpha_p)]\ddot{\alpha}_p \\
&+ m_s L\cos(\alpha_t - \alpha_p)\ddot{d}_{pz} + m_s L^2\ddot{\theta} \\
&+ m_s L[d_{pz}\sin(\alpha_t - \alpha_p) + d_{py}\cos(\alpha_t - \alpha_p)]\ddot{\theta} \\
&+ m_s L(\dot{\alpha}_p + \dot{\theta})^2[d_{py}\sin(\alpha_t - \alpha_p) - d_{pz}\cos(\alpha_t - \alpha_p)] \\
&+ 2m_s L\dot{d}_{pz}(\dot{\alpha}_p + \dot{\theta})\sin(\alpha_t - \alpha_p) \\
&+ \frac{\mathrm{GM}}{2R_c^3}\{-2m_s L[d_{py}\sin(\alpha_t - \alpha_p) - d_{pz}\cos(\alpha_t - \alpha_p)] \\
&+ 6m_s L\sin(\alpha_t)[d_{py}\cos(\alpha_p) - d_{pz}\sin(\alpha_p)] \\
&+ 3m_s L^2\sin(2\alpha_t)\} = 0
\end{aligned}$$

In the present model, the platform pitch (α_p) response is controlled by a momentum wheel located at its c.m., and the offset control strategy is used to control the tether swing. For the offset control strategy the acceleration of the tether attachment point, $\ddot{d}_{pz}$, is considered as the input to the system. In this case it is important to control the offset position, i.e., the distance between the tether hinge point and the c.m. of the platform. This needs to augment the system dynamics by the equation

$$\ddot{d}_{pz} = u_{\alpha_t}$$

so that the inputs M_x and u_{α_t} can be selected to control α_p, α_t, and d_{pz}. The equations of motion mentioned above are linearized about the zero operating point. For a 500-m tether and the inertial parameters as mentioned before, the linear system dynamics can be represented by the matrices

$$A = \begin{bmatrix}
0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \\
-6.2213e-07 & 2.3950e-08 & -9.2303e-10 & 0.0 & 0.0 & 0.0 \\
2.5459e-08 & -3.4255e-06 & 3.7772e-11 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0
\end{bmatrix}$$

$$B = \begin{bmatrix}
0.0 & 0.0 \\
0.0 & 0.0 \\
0.0 & 0.0 \\
4.9999e-09 & -1.4777e-06 \\
-2.0460e-10 & -2.0460e-03 \\
0.0 & 1.0
\end{bmatrix}$$

The above system matrix A corresponds to a state vector given by

$$x = \{\alpha_p \quad \alpha_t \quad d_{pz} \quad \dot{\alpha}_p \quad \dot{\alpha}_t \quad \dot{d}_{pz}\}^T$$

and the control influence matrix B is for two inputs, M_x and u_{α_t} ($= \ddot{d}_{pz}$).

Controller Design and its Implementation

Design of State Feedback Controller

The linear state space model (A, B) obtained in the previous section is found to be controllable, and hence it is possible to assign arbitrary eigenvalues to the system. Furthermore, since the rank of the matrix B is 2, only two elements of each eigenvector can be assigned.

The major difficulty in the controller design is the selection of desired eigenvalues and eigenvectors. There is no closed-form method available in the literature to translate the specifications into the desired eigenvalues and eigenvectors for systems having order greater than 2. Normally, this is done by trial and error. In the present problem, the first requirement is to decouple the platform from tether and offset dynamics. Keeping this in mind, the structure of the closed-loop eigenvectors can be selected as

$$V = [v_1^d \quad v_2^d \quad v_3^d \quad v_4^d \quad v_5^d \quad v_6^d] = \begin{bmatrix} 1 & x & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & y & y_1 & y_2 \\ 0 & 0 & y & 1 & z_1 & z_2 \\ 0 & 0 & y_1 & z_1 & 1 & z \\ 0 & 0 & y_2 & z_2 & z & 1 \end{bmatrix}$$

The above selection of eigenvectors may not be the best possible choice. In fact, the choice of the eigenvectors depends on the designer's requirement.

The numerical values for the entries of the eigenvectors and eigenvalues are selected by trial and error so that the settling time for all the degrees of freedom is less than 0.5 orbits and the offset position is within ±20 m for an initial disturbance of 5 deg in platform and tether pitch. The values considered for the controller design are

$$\lambda = \begin{Bmatrix} -0.00095 \pm 0.0015j \\ -0.00094 \pm 0.0013j \\ -0.0018 \pm 0.0011j \end{Bmatrix}$$

$$x = 2.0, \qquad y = 0.2, \qquad z = 2.0$$

$$y_1 = 2.0, \qquad y_2 = 2.0$$

$$z_1 = 0.8, \qquad z_2 = 0.8$$

Knowing the desired closed-loop eigenvectors and eigenvalues, the controller gain matrix can be obtained. In the present case the B matrix has condition number of $2.0e+8$, and the matrix product $B^T B$ is singular. To overcome this singularity problem, the control input vector is modified and the scaling matrix used,

$$M = \begin{bmatrix} 1.0e-6 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$

With this scaling, the condition number of the modified control influence matrix becomes 200.0004. After translating the controller to correspond to the original control input, the feedback gain matrix becomes

$$K = \begin{bmatrix} -5.3091e+2 & 1.8911e+2 & -8.8104e-1 & -3.9777e+5 & -3.5966e+5 & -7.8589e+2 \\ 4.2715e-6 & 3.3999e-3 & -3.2106e-6 & -6.5015e-2 & 2.0544e-1 & -5.0445e-3 \end{bmatrix}$$

The matrix operations were carried out using MATLAB.

Implementation of the Controller

The state feedback controller, designed based on the rigid-body model, is implemented on the complete nonlinear and flexible model. There are two objectives: to validate the effectiveness of the controller through simulation and to study the effect of the controller, designed for rigid degrees of freedom, on the flexible dynamics of the tether. As expected, the rigid-body dynamics is controlled quite well with settling time less than 0.4 orbit (for an error limit of ±5% of the initial conditions) for both platform response α_p and tether oscillation α_t (Fig. 3). As can be observed from the figure, the tether flexibility does not have any significant effect on the rigid-body dynamics and shows stable oscillatory behavior. Since the rigid-body dynamics is controlled within a period of 0.4 orbit, low-frequency modulations of the longitudinal oscillations, which were observed in the uncontrolled dynamics, are no longer present.

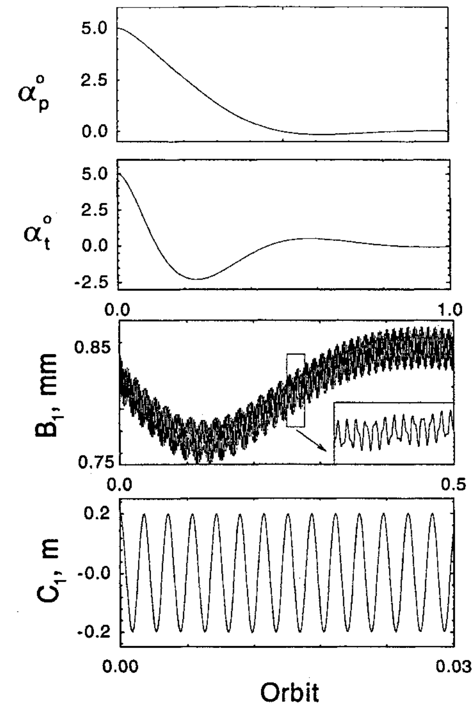

Fig. 3 Controlled response of system with offset strategy to regulate the tether libration.

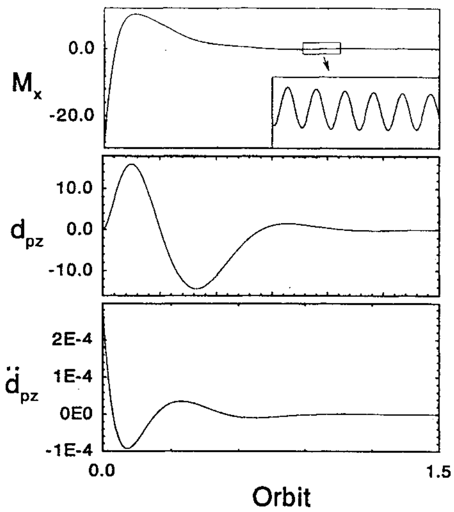

Fig. 4 Time history of offset position and control inputs.

The offset (d_{pz}) motion is within the limit of ±20 m (Fig. 4). Effect of the tether flexibility is also evident in the figure (inset). Since the control inputs are functions of only rigid degrees of freedom, it implies that the rigid-body response is affected by the flexible dynamics. The coupling is as expected because, in the complete equations of motion, the rigid and the flexible generalized coordinates are coupled. However, the coupling is rather weak and remains undetected in the rigid-body response (Fig. 3). Because of the larger gains, the effect of small modulations of α_p, α_t responses can be detected in M_x time history after local magnification. Modulations of the other control input, i.e., $\ddot{d}_{pz}$, was found to be negligible due to small gains and hence could not be detected.

Concluding Remarks

A methodology for the eigenstructure assignment of the LTI multi-input system is developed. The derivation starts with parameterizing the eigenvectors of the system in terms of the controller parameters and closed-loop eigenvalues. Transformation of the closed-loop system states by a similarity relation results in an expression for the controller gains. While defining the similarity transformation matrix, difficulties arise if some of the desired closed-loop eigenvalues are the same as those of the open-loop system. However, the problem is readily overcome by assigning negligible shift to the values. An example involving control of TSS shows that the eigenstructure can be assigned with a considerable degree of accuracy.

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